

COBORDISM OF MAPS ON \mathbb{Z}_2 -WITT SPACES

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ABSTRACT. In this article we study the bordism groups of normally nonsingular maps $f : X \rightarrow Y$ defined on pseudomanifolds X and Y . To characterize the bordism of such maps, inspired by the formula given by Stong, we give a general definition of Stiefel-Whitney numbers defined on X and Y using the Wu classes defined by Goresky and Pardon in [9] and we show that in several cases the cobordism class of a normally nonsingular map $f : X \rightarrow Y$ guarantees that these numbers are zero.

1. INTRODUCTION

The ambient bordism of manifolds was presented by Thom in [14]. Conner and Floyd [4] extended this theory to bordism of maps between closed manifolds and there is a classical work of Stong [13], where the bordism class of maps between manifolds $f : X \rightarrow Y$ is characterized in terms of so-called Stiefel-Whitney numbers of (f, X, Y) .

Concerning the bordism on singular varieties, Siegel in [12] computed the bordism groups of \mathbb{Q} -Witt spaces, showing that in non trivial cases they are equal to the Witt groups. Pardon [11] computed the bordism groups of the “Poincaré duality spaces” defined by Goresky and Siegel in [10].

In this article we extend this notion to bordism groups of normally nonsingular maps $f : X \rightarrow Y$ between pseudomanifolds. For closed smooth manifolds X and Y this definition becomes the Stong’s definition of cobordism of maps (f, X, Y) given in [13].

To characterize the bordism of such maps, inspired by the formula given by Stong, we give a general definition of Stiefel-Whitney numbers defined on X and Y using the Wu classes defined by Goresky and Pardon in [9] and we show that, in several cases the cobordism class of the map f guarantees that these numbers are zero. More precisely, we show how to extend the result of Stong in the case of normally nonsingular maps $f : X \rightarrow Y$ in the following situations: Firstly we consider the case X is a locally orientable \mathbb{Z}_2 -Witt space of pure dimension a and Y an b -dimensional smooth manifold. Then we consider the case X is an a -dimensional smooth manifold and Y a locally orientable \mathbb{Z}_2 -Witt space of pure dimension b . To conclude, we consider the general case where X and Y are locally orientable \mathbb{Z}_2 -Witt spaces.

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2. INTERSECTION HOMOLOGY

2.1. Pseudomanifolds.

Definition 2.1. A pseudomanifold (without boundary) of dimension a is a compact space X which is the closure of the union of $(a - 1)$ -dimensional simplices in any triangulation of X , and each $(a - 1)$ simplex is a face of exactly two a -simplices.

A pseudomanifold (with boundary) of dimension a is a compact space X which is the closure of the union of $(a - 1)$ -dimensional simplices in any triangulation of X , and each $(a - 1)$ simplex is a face of either one or two a -simplices. The boundary consists of simplices which are faces of only one a -simplex.

Every pseudomanifold admits a piecewise linear (P.L. for short) stratification, which is a filtration by closed subspaces $\emptyset \subset X_0 \subset X_1 \subset \dots \subset X_{a-2} \subset X_a = X$, such that for each point $x \in X_i - X_{i-1}$ there is a neighborhood U and a P.L. stratum preserving homeomorphism between U and $\mathbb{R}^{a-i} \times C(L)$, where L is the link of the stratum $X_i - X_{i-1}$ and $C(L)$ denotes the cone on L . Thus, if $X_i - X_{i-1}$ is non empty, it is a (non necessarily connected) manifold of dimension i , and is called the i -dimensional stratum of the stratification.

The singular part, denoted by ΣX , is contained in the element X_{a-2} of the filtration.

Definition 2.2. [5] A map $f : X \rightarrow Y$ between pseudomanifolds is normally nonsingular if there exists a diagram

$$\begin{array}{ccc} N & \xrightarrow{i} & Y \times \mathbb{R}^k \\ \pi \downarrow & \searrow s & \downarrow p \\ X & \xrightarrow{f} & Y \end{array}$$

where $\pi : N \rightarrow X$ is a vector bundle with zero-section s , i is an open embedding, p is the first projection and one has $f = p \circ i \circ s$. The bundle $N = N_f$ is called the normal bundle.

2.2. Intersection Homology and Cohomology. All homology and cohomology groups will be considered with \mathbb{Z}_2 coefficients. Reference for this section is Goresky-MacPherson original paper [7].

The notion of perversity is fundamental for the definition of intersection homology and cohomology. A perversity \bar{p} is a multi-index sequence of integers $(p(2), p(3), \dots)$ such that $p(2) = 0$ and $p(c) \leq p(c + 1) \leq p(c) + 1$, for $c \geq 2$. Any perversity \bar{p} lies between the zero perversity $\bar{0} = (0, 0, 0, \dots)$ and the total perversity $\bar{t} = (0, 1, 2, 3, \dots)$. In particular, we will use the lower middle perversity, denoted \bar{m} and the upper middle perversity, denoted \bar{n} , such that

$$\bar{m}(c) = \left\lfloor \frac{c-2}{2} \right\rfloor \quad \text{and} \quad \bar{n}(c) = \left\lceil \frac{c-1}{2} \right\rceil, \quad \text{for } c \geq 2.$$

Let X be an a -dimensional pseudomanifold and \bar{p} a perversity. The intersection homology groups with \mathbb{Z}_2 coefficients, denoted $IH_i^{\bar{p}}(X)$, are the homology groups of the chain complex

$$IC_i^{\bar{p}}(X) = \left\{ \xi \in C_i(X) \mid \begin{array}{l} \dim(|\xi| \cap X_{a-c}) \leq i - c + p(c) \text{ and} \\ \dim(|\partial \xi| \cap X_{a-c}) \leq i - 1 - c + p(c) \end{array} \right\},$$

where $C_i(X)$ denotes the group of compact i -dimensional P.L. chains ξ of X with \mathbb{Z}_2 coefficients and $|\xi|$ denotes the support of ξ .

In fact $C_*(X)$ is the direct limit $\lim_{\rightarrow} C_*^{\mathcal{T}}(X)$, where $C_*^{\mathcal{T}}(X)$ is the simplicial chain complex with respect to a triangulation \mathcal{T} and the direct limit is taken with respect to subdivision within the family of triangulations of X compatible with the filtration of X .

The intersection cohomology groups with \mathbb{Z}_2 coefficients, denoted $IH_{\bar{p}}^{a-i}(X)$, are defined as the groups of the cochain complex

$$IC_{\bar{p}}^{a-i}(X) = \left\{ \gamma \in C^{a-i}(X) \mid \begin{array}{l} \dim(|\gamma| \cap X_{a-c}) \leq i - c + p(c) \text{ and} \\ \dim(|\partial\gamma| \cap X_{a-c}) \leq i - 1 - c + p(c) \end{array} \right\},$$

where $C^{a-i}(X)$ denotes the abelian group, with \mathbb{Z}_2 coefficients, of all $(a-i)$ -dimensional P.L. cochains of X with closed supports in X .

The main properties of intersection homology that we will use are the following:

For any perversity \bar{p} , the Poincaré map PD, cap-product by the fundamental class of X naturally factorizes in the following way [7]:

$$\begin{array}{ccc} H^{a-i}(X) & \xrightarrow{\quad PD \quad} & H_i(X) \\ & \searrow \alpha_X \quad \nearrow \omega_X & \\ & IH_i^{\bar{p}}(X). & \end{array}$$

where α_X is induced by the cap-product by the fundamental class $[X]$ and ω_X is induced by the inclusion $IC_i^{\bar{p}}(X) \hookrightarrow C_i(X)$.

For perversities \bar{p} and \bar{r} such that $\bar{p} + \bar{r} \leq \bar{t}$, the intersection product

$$IH_i^{\bar{p}}(X) \times IH_j^{\bar{r}}(X) \rightarrow IH_{(i+j)-a}^{\bar{p}+\bar{r}}(X)$$

is well defined.

The natural homomorphism $IH_{\bar{p}}^{a-i}(X) \rightarrow IH_i^{\bar{p}}(X)$, cap-product by the fundamental class $[X]$, is an isomorphism.

3. WITT SPACES AND WU CLASSES

In this section we use definitions and notations of M. Goresky [6] and M. Goresky and W. Pardon [9]. First of all, let us fix notations in the smooth case.

Let X be an a -dimensional manifold. We will denote by $w^i(X) \in H^i(X)$ the Stiefel-Whitney cohomology classes (S-W cohomology classes) of the tangent bundle TX . The Stiefel-Whitney homology classes (S-W homology classes) of TX denoted by $w_{a-i}(X) \in H_{a-i}(X)$ are their images by Poincaré duality. Let $i : X \hookrightarrow V$ be the inclusion of differentiable manifolds, then one has the naturality formula $i^*(w^i(V)) = w^i(X)$.

In the singular case, the *Steenrod square operations* are defined in intersection cohomology by M. Goresky [6, §3.4] as follows:

Definition 3.1. Let X be an a -dimensional pseudomanifold. Suppose \bar{c} and \bar{d} are perversities such that $2\bar{c} \leq \bar{d}$. For any i with $0 \leq i \leq [a/2]$ the “Steenrod square” operation

$$Sq^i : IH_{\bar{c}}^j(X) \rightarrow IH_{\bar{d}}^{i+j}(X) \rightarrow \mathbb{Z}_2$$

is given by multiplication with the intersection cohomology i^{th} -Wu class of X :

$$v^i(X) = v_{\bar{d}}^i(X) \in IH_{\bar{d}}^i(X).$$

One defines $v^i(X) = 0$, for $i > [a/2]$.

Definition 3.2. ([9], Definition 10.1) A stratified pseudomanifold X is a \mathbb{Z}_2 -Witt space if for each stratum of odd codimension $2k+1$, $IH_k^{\bar{n}}(L) = 0$, where L is the link of the stratum.

For such spaces, the middle intersection homology group satisfies the Poincaré duality over \mathbb{Z}_2 .

In the following, we will use the notion of *locally orientable Witt-space* that we recall:

Definition 3.3. ([9], Definition 10.2) A stratified pseudomanifold X is a locally orientable Witt space if it is both locally orientable and a \mathbb{Z}_2 -Witt space.

Let X be a \mathbb{Z}_2 -Witt space, then the Wu classes $v^i(X)$ lift canonically to $IH_m^i(X) = IH_n^i(X)$ (see [9] §10). We denote by $v_{a-i}(X) \in IH_{a-i}^{\bar{n}}(X)$ the (homology) $(a-i)^{th}$ -Wu class of X , in intersection homology, dual to the Wu class $v^i(X)$ (denoted by $Iv^i \in IH^i(X)$ in [6]).

Definition 3.4. [6, 12] One defines the Whitney classes by

$$IW_{a-i}(X) = \sum_{\ell+j=i} Sq^\ell v^j(X) \in IH_t^i(X) = H_{a-i}(X).$$

The pullback of the intersection cohomology Whitney class under a normally nonsingular map is given by the following theorem ([6, 5.3]):

Theorem 3.5. Let X and Y be \mathbb{Z}_2 -Witt spaces and $f : X \rightarrow Y$ a normally nonsingular map with normal bundle N_f . Then one has, in $IH^*(X)$:

$$f^*(IW(Y)) = W(N_f) \cup IW(X)$$

where $W(N_f)$ is the Whitney cohomology class (in $H^*(X)$) of the normal bundle N_f .

The inclusion $j : X \hookrightarrow V$ provides an unique morphism $j^* : IH_q^{n-i}(V) \rightarrow IH_q^{n-i}(X)$ (see [3] §(3.4)). The result comes from the commutative diagram

$$\begin{array}{ccc} IH_q^{n-i}(X) & \xleftarrow{j^*} & IH_q^{n-i}(V) \\ \uparrow \cong & & \uparrow \cong \\ IH_i^{\bar{p}}(X) & \xleftarrow{j_X^*} & IH_{i+1}^{\bar{p}}(V) \end{array}$$

where the bottom map j_X^* is defined by the upper one. We have:

Corollary 3.6. Let us consider the inclusion $j : X \hookrightarrow V$ of the \mathbb{Z}_2 -Witt space X in a \mathbb{Z}_2 -Witt space V such that $\Sigma V \subset X$, so that the normal bundle N_i is trivial. Then one has:

$$j_X^*(v_{i+1}(V)) = v_i(X).$$

4. COBORDISM OF MAPS

Definition 4.1. Let $f : X \rightarrow Y$ be a normally nonsingular map between pseudomanifolds of dimensions a and b respectively. The triple (f, X, Y) bords if there exist:

- (1) Pseudomanifolds V and W with dimensions $a+1$ and $b+1$, respectively, such that $\partial V = X$ and $\partial W = Y$; $\Sigma V \subset X$ and $\Sigma W \subset Y$.
- (2) $F : V \rightarrow W$ normally nonsingular such that $F|_X = f$.

We will denote $(f, X, Y) = \partial(F, V, W)$.

The definition implies that $V \setminus X$ and $W \setminus Y$ are smooth manifolds. If X (resp. Y) is a manifold, then W (resp. V) is a manifold with smooth boundary.

If we consider X and Y closed smooth manifolds, this definition becomes the Stong's definition to cobordism of maps (f, X, Y) in [13]. In this case Stong defines the Stiefel-Whitney (S-W for short) numbers associated to the map (f, X, Y) ; these numbers allow to characterize the bordism properties among such maps. We recover here results described by Stong which are necessary to better understand our main results.

Definition 4.2. [13] Let us consider a map $f : X \rightarrow Y$, where X and Y are manifolds of dimensions a and b , respectively. Define $f^! : H^i(X) \rightarrow H^{i+b-a}(Y)$ in such a way that for any $\alpha \in H^i(X)$, we define $f^!(\alpha) : H_{i+b-a}(Y) \rightarrow \mathbb{Z}_2$ such that for each $\beta \in H_{i+b-a}(Y)$,

$$f^!(\alpha)(\beta) = \langle f^*(\tilde{\beta}) \cup \alpha, [X] \rangle \in \mathbb{Z}_2,$$

where $\tilde{\beta} \in H^{a-i}(Y)$ is the Poincaré dual of β .

Remark 4.3. According to Atiyah and Hirzebruch [1], the map $f^!$ can be described in the following way: let us consider $h : X \rightarrow \mathbb{S}^s$ an imbedding of X in some s -dimensional sphere \mathbb{S}^s and T a tubular neighborhood of $(f \times h)(X)$ in $Y \times \mathbb{S}^s$, then $f^!$ is the composition of the maps:

$$H^i(X) \xrightarrow{\varphi} H^{i+s+b-a}(T/\partial T) \xrightarrow{c^*} H^{i+s+b-a}(Y \times \mathbb{S}^s) \simeq H^{i+b-a}(Y),$$

where φ denotes the Thom isomorphism and $c : Y \times \mathbb{S}^s \rightarrow T/\partial T$ is the contraction.

5. MAIN RESULTS

In this section we show how to extend the result in the case of singular spaces and normally nonsingular maps $f : X \rightarrow Y$. Firstly we consider the case X is a locally orientable \mathbb{Z}_2 -Witt space of pure dimension a and Y a b -dimensional smooth manifold. Then we consider the case X is an a -dimensional smooth manifold and Y is a locally orientable \mathbb{Z}_2 -Witt space of pure dimension b . To conclude, we consider the general case where X and Y are locally orientable \mathbb{Z}_2 -Witt spaces.

5.1. Case of a map $f : X \rightarrow Y$, with Y a smooth manifold.

Let $f : X \rightarrow Y$ be a normally nonsingular map, with X a locally orientable \mathbb{Z}_2 -Witt space of pure dimension a and Y a b -dimensional smooth manifold.

Definition 5.1. Let us define the map $f_B : IH_i^{\bar{p}}(X) \rightarrow IH_i^{\bar{p}}(Y)$ in such a way that the following diagram commutes

$$\begin{array}{ccc} H_i(X) & \xrightarrow{f_*} & H_i(Y) \\ \uparrow \omega_X & & \uparrow \omega_Y \simeq \\ IH_i^{\bar{p}}(X) & \xrightarrow{f_B} & IH_i^{\bar{p}}(Y) \end{array}$$

i.e. $f_B = (\omega_Y)^{-1} \circ f_* \circ \omega_X$, where the map ω_Y is an isomorphism since Y is smooth.

We denote by \tilde{f}_B the map obtained by composition

$$IH_i^{\bar{p}}(X) \xrightarrow{\omega_X} H_i(X) \xrightarrow{f_*} H_i(Y) \xrightarrow{PD} H^{b-i}(Y)$$

with Poincaré duality PD .

Definition 5.2. For any partition $\iota = (\iota_1, \dots, \iota_s)$ and r numbers u_1, \dots, u_r satisfying

$$(5.3) \quad (\iota_1 + \dots + \iota_s) + u_1 + \dots + u_r + r(b-a) = b,$$

let us denote $w^\iota(Y) = w^{\iota_1}(Y) \cdots w^{\iota_s}(Y)$. The S-W numbers of any triple (f, X, Y) are defined by

$$\langle w^\iota(Y) \cdot \tilde{f}_B(v_{a-u_1}(X)) \cdots \tilde{f}_B(v_{a-u_r}(X)), [Y] \rangle.$$

Theorem 5.4. Let $f : X \rightarrow Y$ be a normally nonsingular map, with X a locally orientable \mathbb{Z}_2 -Witt space of pure dimension a and Y a b -dimensional smooth manifold. If (f, X, Y) bords, then for any partition ι and r numbers u_1, \dots, u_r satisfying (5.3), the S-W numbers

$$\langle w^\iota(Y) \cdot \tilde{f}_B(v_{a-u_1}(X)) \cdots \tilde{f}_B(v_{a-u_r}(X)), [Y] \rangle$$

are zero.

Proof. As (f, X, Y) bords, one has $(f, X, Y) = \partial(F, V, W)$. We may define a map

$$\tilde{F}_B : IH_i^{\bar{p}}(V) \rightarrow IH_i^{\bar{p}}(W) = H_i(W) \rightarrow H^{b+1-i}(W)$$

in the same way that we defined \tilde{f}_B .

One has:

$$\langle w^\iota(Y) \cdot \tilde{f}_B(v_{a-u_1}(X)) \cdots \tilde{f}_B(v_{a-u_r}(X)), [Y] \rangle =$$

$$\langle j^* w^\iota(W) \cdot j^* \tilde{F}_B(v_{a-u_1}(V)) \cdots j^* \tilde{F}_B(v_{a-u_r}(V)), \partial[W] \rangle,$$

by corollary 3.6 and commutativity of the following diagram:

$$\begin{array}{ccccc}
 H_{i+1}(V) & \xrightarrow{F_*} & H_{i+1}(W) & & \\
 \uparrow \omega_V & \searrow \tilde{j} & \swarrow \tilde{j} & & \uparrow \\
 & H_i(X) & \xrightarrow{f_*} & H_i(Y) & \\
 & \uparrow \omega_X & & \uparrow PD \cong & \\
 & IH_i^{\bar{p}}(X) & \xrightarrow{\tilde{f}_B} & H^{b-i}(Y) & \\
 & \swarrow j_X^* & & \swarrow j^* & \\
 IH_{i+1}(V) & \xrightarrow{\tilde{F}_B} & H^{b-i}(W) & &
 \end{array}$$

So, we obtain:

$$\begin{aligned}
 & \left\langle j^* \left(w^t(W) \cdot \tilde{F}_B(v_{a-u_1}(V)) \cdots \tilde{F}_B(v_{a-u_r}(V)) \right), \partial[W] \right\rangle = \\
 & \left\langle \delta j^* \left(w^t(W) \cdot \tilde{F}_B(v_{a-u_1}(V)) \cdots \tilde{F}_B(v_{a-u_r}(V)) \right), [W, \partial W] \right\rangle = 0,
 \end{aligned}$$

where

$$H^k(W) \xrightarrow{j^*} H^k(Y) \xrightarrow{\delta} H^{k+1}(W, \partial W)$$

is part of a long exact sequence, so that $\delta j^* = 0$. \square

5.2. Case of a map $f : X \rightarrow Y$, with X a smooth manifold.

Let $f : X \rightarrow Y$ be a normally nonsingular map, with X an a -dimensional smooth manifold and Y a locally orientable \mathbb{Z}_2 -Witt space of pure dimension b .

Since f is a normally nonsingular map one may consider the normal bundle N_f over X , and $i : N_f \rightarrow Y \times \mathbb{R}^{s+1}$ an open imbedding. Let \tilde{T} be a tubular neighborhood of $(f \times h)(X)$ in $Y \times \mathbb{R}^{s+1}$, where $h : X \rightarrow \mathbb{R}^{s+1}$ is defined in such a way that the following diagram commutes.

$$\begin{array}{ccc}
 N_f & \xrightarrow{i} & Y \times \mathbb{R}^{s+1} \\
 \uparrow \sigma & \nearrow f \times h & \\
 X & &
 \end{array}$$

We denote by \mathbb{S}^s the s -dimensional sphere in \mathbb{R}^{s+1} and by T the intersection $T = \tilde{T} \cap (Y \times \mathbb{S}^s)$. Following the remark 4.3, there exists a map ϕ which is the composition of the maps:

$$H^i(X) \xrightarrow{\varphi} H^{i+s+b-a}(T/\partial T) \xrightarrow{c^*} H^{i+s+b-a}(Y \times \mathbb{S}^s) \simeq H^{i+b-a}(Y),$$

here φ denotes the Thom homomorphism and $c : Y \times \mathbb{S}^s \rightarrow T/\partial T$ is the contraction. The last isomorphism is given by the *Künneth formula* for a product of a smooth manifold with a \mathbb{Z}_2 -Witt space [3].

Since X is a smooth manifold, $\alpha_X : H^i(X) \rightarrow IH_{a-i}^{\bar{p}}(X)$ is an isomorphism, then one defines the map f_B by commutativity of the following diagram, *i.e.* as being $f_B = \alpha_Y \circ \phi \circ \alpha_X^{-1}$

$$\begin{array}{ccc} H^i(X) & \xrightarrow{\phi} & H^{b-(a-i)}(Y) \\ \alpha_X \downarrow \cong & & \downarrow \alpha_Y \\ IH_{a-i}^{\bar{p}}(X) & \xrightarrow{f_B} & IH_{a-i}^{\bar{p}}(Y). \end{array}$$

For any u with $0 \leq u \leq b$, let $v_u(Y) \in IH_u^{\bar{m}}(Y)$ the Wu class of Y , dual of $v^{b-u}(Y) \in IH_{\bar{m}}^{b-u}(Y)$ and $w_{b-u}(X)$ the homology Whitney class of X , so that $f_B(w_{b-u}(X)) \in IH_{b-u}^{\bar{m}}(Y)$. For any u with $0 \leq u \leq b$ the S-W intersection numbers

$$v_u(Y) \cdot f_B(w_{b-u}(X))$$

are well defined.

Theorem 5.5. Let $f : X \rightarrow Y$ be a normally nonsingular map, with X an a -dimensional smooth manifold and Y a locally orientable \mathbb{Z}_2 -Witt space of pure dimension b . If (f, X, Y) bords, then for any $0 \leq u \leq b$ the S-W numbers

$$v_u(Y) \cdot f_B(w_{b-u}(X))$$

are zero.

Proof. If $(f, X, Y) = \partial(F, V, W)$, one has

$$H^i(V) \xrightarrow{\varphi} H^{i+s+b-a}(T'/\partial T') \xrightarrow{c^*} H^{i+s+b-a}(W \times \mathbb{S}^s) \simeq H^{i+b-a}(W),$$

where V is embedded in \mathbb{S}^s and T' is a tubular neighborhood of $(F \times h)(V)$, which gives rise to the corresponding map F_B . Therefore we can consider the following diagram, where PD denotes the Poincaré duality

$$\begin{array}{ccccc} & H_{a-i}(X) & \xrightarrow{f_*} & H_{a-i}(Y) & \\ & \uparrow \omega_X & & \uparrow \omega_Y & \\ & IH_{a-i}^{\bar{p}}(X) & \xrightarrow{f_B} & IH_{a-i}^{\bar{p}}(Y) & \\ \nearrow \alpha_X & & & & \nwarrow \alpha_Y \\ H^i(X) & \xrightarrow{\varphi} & H^{i+s+b-a}(T/\partial T) & \xrightarrow{c^*} & H^{i+s+b-a}(Y \times \mathbb{S}^s) \simeq H^{i+b-a}(Y) \end{array}$$

since we had defined f_B and F_B the result follows in the same way of the proof of Theorem 5.4. \square

5.3. The general case.

In the general case X and Y are locally orientable \mathbb{Z}_2 -Witt spaces of dimensions a and b respectively. It is not always possible to define an unique map f_B as done in the other cases, however we can show that for any map \tilde{f}_B considered, the bordism condition of (f, X, Y) implies that the corresponding S-W numbers are zero.

First we show the following lemma.

Lemma 5.6. Let $f : X \longrightarrow Y$ be a normally nonsingular map and $(f, X, Y) = \partial(F, V, W)$. Given a map f_B there exists a map F_B such that the following diagram commutes.

$$\begin{array}{ccc} IH_u^{\bar{m}}(X) & \xleftarrow{j_X^*} & IH_{u+1}^{\bar{m}}(V) \\ f_B \downarrow & & \downarrow F_B \\ IH_u^{\bar{m}}(Y) & \xleftarrow{j_Y^*} & IH_{u+1}^{\bar{m}}(W). \end{array}$$

Proof. The diagram

$$\begin{array}{ccc} X & \xrightarrow{j_X} & V \\ f \downarrow & & \downarrow F \\ Y & \xrightarrow{j_Y} & W \end{array}$$

is a cartesian diagram. Then we can apply Proposition 10.7 in [2] (see also [8]).

One has equality of sheaves on Y :

$$j_Y^* F_! \mathcal{A} = f_! j_X^* \mathcal{A}$$

for any sheaf \mathcal{A} on V . That provides a commutative diagram of complexes of sheaves on Y (perverse intersection sheaves for the middle perversity \bar{m}).

$$\begin{array}{ccc} f_! \mathcal{IC}_X^\bullet & \longleftarrow & f_! j_X^* (\mathcal{IC}_V^\bullet) = j_Y^* F_! (\mathcal{IC}_V^\bullet) \\ \downarrow & & \downarrow \\ \mathcal{IC}_Y^\bullet & \longleftarrow & j_Y^* (\mathcal{IC}_W^\bullet) \end{array}$$

Let us remind that intersection homology is obtained by taking hypercohomology of the perverse intersection sheaf:

$$IH_u^{\bar{m}}(Y) = \mathbb{H}^{b-u}(Y; \mathcal{IC}_Y^\bullet)$$

Taking hypercohomology

$$\mathbb{H}^{b-u}(Y; \bullet)$$

of the previous diagram, one obtains:

$$\begin{array}{ccc} \mathbb{H}^{b-u}(X; \mathcal{IC}_X^\bullet) & \xleftarrow{j_X^*} & \mathbb{H}^{b-u}(V; \mathcal{IC}_V^\bullet) \\ f_B \downarrow & & \downarrow F_B \\ \mathbb{H}^{b-u}(Y; \mathcal{IC}_Y^\bullet) & \xleftarrow{j_Y^*} & \mathbb{H}^{b-u}(W; \mathcal{IC}_W^\bullet) \end{array}$$

and the Lemma follows. \square

Theorem 5.7. Let $f : X \longrightarrow Y$ be a normally nonsingular map, with X and Y locally orientable \mathbb{Z}_2 -Witt spaces of pure dimension a and b respectively. Then for any u with $0 \leq u \leq b$, the S-W numbers $\langle v_u(Y), f_B(v_{b-u}(X)), [Y] \rangle$ are zero.

Proof. The diagram of Lemma 5.6 can be written in the cohomology setting

$$\begin{array}{ccc} IH_{\bar{n}}^{a-u}(X) & \xrightarrow{f^B} & IH_{\bar{n}}^{b-u}(Y) \\ \uparrow j_X^* & & \uparrow j_Y^* \\ IH_{\bar{n}}^{a-u}(V) & \xrightarrow{F^B} & IH_{\bar{n}}^{b-u}(W). \end{array}$$

where $\bar{m} + \bar{n} = \bar{t}$ and we use the same notation for corresponding maps j_X^* and j_Y^* .

Let us consider the homology class $v_{b-u}(Y) \in IH_{b-u}^{\bar{n}}(Y)$, that will be written $v^u(Y) \in IH_{\bar{m}}^u(Y)$ in the cohomology setting.

Then $v^u(Y) = j_Y^* v^u(W)$ where $v^u(W) \in IH_{\bar{m}}^u(W)$ is the corresponding cohomology Wu class to the homology Wu class $v_{b+1-u}(W) \in IH_{b+1-u}^{\bar{n}}(W)$ of W .

Let us consider the cohomology Wu class $v^{a-u}(X) \in IH_{\bar{n}}^{a-u}(X)$ corresponding for the cohomology Wu class $v_u(X) \in IH_u^{\bar{m}}(X)$. Then $v^{a-u}(X) = j_X^*(v^{a-u}(V))$ where $v^{a-u}(V) \in IH_{\bar{n}}^{a-u}(V)$ is the corresponding cohomology Wu class to the homology class $v_{u+1}(V) \in IH_{u+1}^{\bar{m}}(V)$.

One has

$$f^B(v^{a-u}(X)) = f^B j_X^*(v^{a-u}(V)) \in IH_{\bar{n}}^{b-u}(Y).$$

The intersection product

$$v_{b-u}(Y) \cdot f_B(v_u(X)) \in IH_{b-u}^{\bar{n}}(Y) \times IH_u^{\bar{m}}(Y) \rightarrow IH_0^{\bar{t}}(Y)$$

corresponds to the product

$$v^u(Y) \cup f^B(v^{a-u}(X)) \in IH_{\bar{m}}^u(Y) \times IH_{\bar{n}}^{b-u}(Y) \rightarrow IH_0^b(Y).$$

One has

$$\langle v^u(Y) \cup f^B(v^{a-u}(X)), [Y] \rangle =$$

$$\langle j_Y^* v^u(W) \cup f^B j_X^*(v^{a-u}(V)), [Y] \rangle =$$

$$\langle j_Y^* v^u(W) \cup j_Y^* F^B(v^{a-u}(V)), [Y] \rangle =$$

$$\langle j_Y^* [v^u(W) \cup F^B(v^{a-u}(V))], [\partial W] \rangle =$$

$$\langle \delta j_Y^* [v^u(W) \cup F^B(v^{a-u}(V))], [W, \partial W] \rangle = 0$$

where the first equality is a consequence of the Theorem 5.3 of Goresky [6], the second one is from Lemma 5.6 and the fourth equality is obtained in an analogous way than the proof of Theorem 5.5. \square

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